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ON THE LINEAR INDEPENDENCE OF CERTAIN COHOMOLOGY CLASSES IN THE CLASSIFYING SPACE FOR SUBFOLIATIONS

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ABSTRACT. The purpose of this paper is to establish the linear independence of certain cohomology classes in the Haefliger classifying space $B\Gamma_{(q_1, q_2)}$ for subfoliations of codimension (q_1, q_2). The classes considered are of secondary type, not belonging to the subalgebra of $H(B\Gamma_{(q_1, q_2)}, R)$ generated by the union of the universal characteristic classes for foliations of codimension q_1 and q_2 respectively, and are elements of the kernel of the canonical homomorphism $H(B\Gamma_{(q_1, q_2)}, R) \to H(B\Gamma_{q_1} \times B\Gamma_d, R)$ with $d = q_2 - q_1 > 0$.

1. INTRODUCTION

Let M be an n-dimensional manifold and TM its tangent bundle. A (q_1, q_2)-codimensional subfoliation on M is a couple (F_1, F_2) of integrable subbundles F_i of TM of dimension $n - q_i$, $i = 1, 2$, with F_2 being at the same time a subbundle of F_1. Therefore, for each $i = 1, 2$, F_i defines a q_i-codimensional foliation on M, $d = q_2 - q_1 \geq 0$, and the leaves of F_1 contain those of F_2, Moussu [18], Feigin [9], and Cordero-Masa [5] have studied the (exotic) characteristic homomorphism of a subfoliation (F_1, F_2), and Carballés [3] has generalized Cordero-Masa's construction, introducing the characteristic homomorphism $\Delta_*(P)\colon H(W(g, H)_I) \to H_{\mathrm{DR}}(M)$ of an (F_1, F_2)-foliated principal bundle $P = P_1 + P_2$ over M of structure group $G = G_1 \times G_2$. The author has computed in [6] the cohomology algebras $H(W(g, H)_I)$ and has given some geometric interpretations for the Godbillon-Vey classes of a subfoliation. Finally, the author has evaluated in [7] the characteristic homomorphism of a subfoliation for the particular case of locally homogeneous subfoliations, using the techniques of Kamber-Tondeur [16] and Carballés [3], and has given several examples of such subfoliations with nontrivial secondary or exotic characteristic classes which do not belong to the subalgebra of $H_{\mathrm{DR}}(M)$ generated by the characteristic classes of the two foliations.

In this paper, using some of the results obtained in [6, 7], we prove the linear independence of certain cohomology classes in the Haefliger classifying space $B\Gamma_{(q_1, q_2)}$ for subfoliations of codimension $(q_1, q_2) = (d+1, 2d+1)$ with $d \geq 1$. These classes are universal secondary characteristic classes for subfoliations of codimension (q_1, q_2), not belonging to the subalgebra of $H(B\Gamma_{(q_1, q_2)}, R)$ generated by the union of the universal characteristic classes for foliations of

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codimension q_1 and q_2 respectively. Furthermore, the classes considered are in the kernel of the canonical homomorphism

$$H(B\Gamma_{(q_1, q_2)}, R) \to H(B\Gamma_{q_1} \times B\Gamma_d, R),$$

where $B\Gamma_{q_1}$ (resp. $B\Gamma_d$) is the Haefliger classifying space for foliations of codimension q_1 (resp. d). A similar result holds for cohomology classes in the Haefliger classifying space $B\Gamma^+_{(q_1, q_2)}$ (resp. $F\Gamma_{(q_1, q_2)}$) for subfoliations of codimension (q_1, q_2) with oriented normal bundle (resp. with trivialized normal bundle).

The paper is structured as follows. In §2 we define the classifying space $B\Gamma_{(q_1, q_2)}$ for subfoliations of codimension (q_1, q_2). In §3 we introduce the universal characteristic homomorphism $\Delta_*: H(WO_I) \to H(B\Gamma_{(q_1, q_2)}, R)$ for subfoliations of codimension (q_1, q_2) using the techniques of Bott [2] and Cordero-Masa [5]. §4 is devoted to the computation of the canonical homomorphism $p_*: H(WO_I) \to H(WO_{q_1}) \otimes H(WO_d)$ using the author's techniques [6]. Finally, in §5 the results obtained in §§3, 4 and [7] are used in order to establish the results of the preceding paragraph.

Throughout this paper all manifolds, foliations, and subfoliations are of type C^∞, and cohomology groups are taken with real coefficients. We also adopt the notations of [3, 6, 7, 16].

2. The classifying space for subfoliations

In this section we define a classifying space for subfoliations. For this purpose, the techniques used by Haefliger in [12] will be adopted here.

Let (q_1, q_2) be a couple of integers q_i with $0 \leq q_1 \leq q_2$. Consider the subgroupoid $\Gamma = \Gamma_{(q_1, q_2)} \subset \Gamma_{q_2}$ of germs of local diffeomorphisms of $R^{q_2} = R^{q_1} \times R^d$ preserving the foliation on R^{q_2} defined by the canonical projection into R^{q_1}, where $d = q_2 - q_1 \geq 0$. Let $G = \mathrm{GL}(q_1, q_2) \subset \mathrm{GL}(q_2)$ be the subgroup of matrices of the form

$$\left(\begin{array}{c|c} A & 0 \\ \hline * & B \end{array}\right)$$

with $A \in \mathrm{GL}(q_1)$ and $B \in \mathrm{GL}(d)$. Clearly, $\mathrm{GL}(q_1) \times \mathrm{GL}(d) \subset G$ is a deformation retract. The differential defines a continuous homomorphism $\Gamma \to G$, and it induces a continuous map $\nu: B\Gamma \to BG$ classifying the normal bundle of the universal Γ-structure on the Haefliger classifying space $B\Gamma$ for Γ.

Now, let (F_1, F_2) be a (q_1, q_2)-codimensional subfoliation on an n-dimensional manifold M. Then (F_1, F_2) is given by an open cover of M by coordinate charts $\{U_\alpha\}_{\alpha\in\Lambda}$ with local coordinate functions $x_1^\alpha, \dots, x_n^\alpha$ satisfying

$$\frac{\partial x_i^\beta}{\partial x_a^\alpha} = \frac{\partial x_i^\beta}{\partial x_u^\alpha} = \frac{\partial x_a^\beta}{\partial x_u^\alpha} = 0 \quad \text{on } U_\alpha \cap U_\beta \text{ for } 1 \leq i \leq q_1 < a \leq q_2 < u \leq n.$$

It follows that there is an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of M and submersions $f_\alpha: U_\alpha \to R^{q_2} = R^{q_1} \times R^d$ such that

$$(F_1|_{U_\alpha}, F_2|_{U_\alpha}) = (f_\alpha^{-1}(0 \times TR^d), f_\alpha^{-1}(0)) \quad \text{for } \alpha \in \Lambda.$$

Hence, using the techniques of Haefliger [12], we obtain the following result.

Proposition 2.1. *A* (q_1, q_2)*-codimensional subfoliation* (F_1, F_2) *on* M *is classified (up to homotopy) by a continuous map* $f: M \to B\Gamma$, *and its normal bundle* $\nu(F_1, F_2) = \nu F_1 \oplus (F_1/F_2)$ *is classified by the composition*

$$M \xrightarrow{f} B\Gamma \xrightarrow{\nu} BG \approx B(\mathrm{GL}(q_1) \times \mathrm{GL}(d)) \\ \approx B\,\mathrm{GL}(q_1) \times B\,\mathrm{GL}(d) \approx B\mathrm{O}(q_1) \times B\mathrm{O}(d).$$

Remarks. (1) It is clear that F_i is classified by the map $f_i = \phi_i \circ f: M \to B\Gamma \to B\Gamma_{q_i}$, $i = 1, 2$, where ϕ_i is the canonical map. Of course, the map $\nu \circ f: M \to B\Gamma \to BG$ classifies the vector bundle $\nu F_2 \cong \nu(F_1, F_2)$ with structure group $G = \mathrm{GL}(q_1, q_2)$.

(2) Similarly, a double foliation on M given by two transverse foliations F_1 and F_0 of codimension q_1 and d respectively is classified (up to homotopy) by a continuous map $f: M \to B(\Gamma_{q_1} \times \Gamma_d) \approx B\Gamma_{q_1} \times B\Gamma_d$. The (q_1, q_2)-codimensional subfoliation $(F_1, F_2) = (F_1, F_1 \cap F_0)$ on M is classified by the map $\phi \circ f: M \to B(\Gamma_{q_1} \times \Gamma_d) \to B\Gamma$, and F_1 (resp. F_0) is classified by the map $\tilde{\phi}_1 \circ f: M \to B(\Gamma_{q_1} \times \Gamma_d) \to B\Gamma_{q_1}$ (resp. by the map $\tilde{\phi}_0 \circ f: M \to B(\Gamma_{q_1} \times \Gamma_d) \to B\Gamma_d$), where ϕ, $\tilde{\phi}_1$, and $\tilde{\phi}_0$ denote the canonical maps.

(3) In the same way, a (q_1, q_2)-codimensional subfoliation (F_1, F_2) with trivialized normal bundle on M (in the sense of Cordero-Masa [5]) is classified (up to homotopy) by a continuous map $f: M \to F\Gamma$, where $F\Gamma$ denotes the homotopy theoretic fiber of the map $\nu: B\Gamma \to BG$. Analogously, a (q_1, q_2)-codimensional subfoliation (F_1, F_2) with oriented normal bundle on M (in the sense of [6]) is classified (up to homotopy) by a continuous map $f: M \to B\Gamma^+$, where $\Gamma^+ = \Gamma^+_{(q_1, q_2)} \subset \Gamma$ is the subgroupoid of all $\gamma \in \Gamma$ such that the differential of γ belongs to the connected component G_0 of the group G.

3. Characteristic classes of subfoliations

In this section the universal characteristic classes for subfoliations are discussed.

Let

$$\nu^*: H(B\mathrm{O}(q_1), R) \otimes H(B\mathrm{O}(d), R) \cong H(BG, R) \to H(B\Gamma, R)$$

be the homomorphism induced by the map $B\Gamma \xrightarrow{\nu} BG \approx B\mathrm{O}(q_1) \times B\mathrm{O}(d)$ in cohomology. Then, using the techniques of Bott [2], from Theorem 3.9 in [5] we obtain Bott's obstruction theorem for subfoliations (actually, for Γ-structures):

Theorem 3.1. $\nu^*(H^i(B\mathrm{O}(q_1), R) \otimes H^j(B\mathrm{O}(d), R)) = 0$ *if at least one of the inequalities* $i > 2q_1$, $i + j > 2q_2$ *is satisfied.*

Similarly, since the characteristic homomorphism for subfoliations is natural with respect to subfoliation preserving maps, we have the following

Theorem 3.2. *There exists a unique homomorphism*

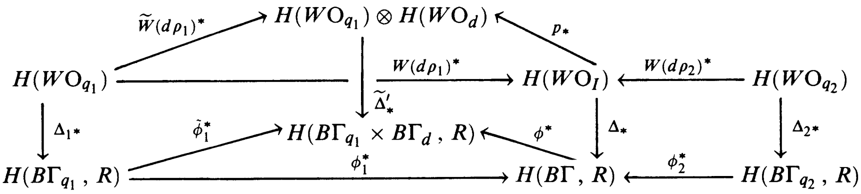
$$\Delta_*: H(W\mathrm{O}_I) \cong H(W(\mathrm{gl}(q_1) \oplus \mathrm{gl}(d), \mathrm{O}(q_1) \times \mathrm{O}(d))_I) \to H(B\Gamma, R)$$

such that $\Delta_{*(F_1, F_2)} = f^* \circ \Delta_*: H(W\mathrm{O}_I) \to H(B\Gamma, R) \to H(M, R) \cong H_{\mathrm{DR}}(M)$ *for any* (q_1, q_2)*-codimensional subfoliation* (F_1, F_2) *on a manifold* M *with classifying map* $f: M \to B\Gamma$, *where* $\Delta_{*(F_1, F_2)}$ *denotes the characteristic homomorphism of* (F_1, F_2) *as defined in* [5].

Definition 3.3. Δ_* is called the *universal characteristic homomorphism* for subfoliations of codimension (q_1, q_2) and the elements of $\operatorname{Im}\Delta_* \subset H(B\Gamma, R)$ are called the *universal characteristic classes* for subfoliations of codimension (q_1, q_2).

Our purpose is to obtain information about $\operatorname{Im}\Delta_* \subset H(B\Gamma, R)$. In the computation of the homomorphism Δ_*, the following result is interesting.

Proposition 3.4. *The diagram*



is commutative, where the vertical maps are the universal characteristic homomorphisms (with $\widetilde{\Delta}'_*$ the universal characteristic homomorphism for double foliations of codimension q_1 and d respectively), and the horizontal maps are the canonical homomorphisms.

Proof. This follows from Theorem 5.2 in [5] and Theorem 4.6 in [6].

Remarks. (1) It is clear that the canonical homomorphisms ϕ_1^*, $\tilde{\phi}_1^*$, $W(d\rho_1)^*$, and $\widetilde{W}(d\rho_1)^*$ are injective, and that $\operatorname{Im}\phi_i^*\Delta_{i*} \subset \operatorname{Im}\Delta_*$, $i = 1, 2$.

(2) For $q_1 = q_2 = q$, and for $q_1 = 0$ and $q_2 = q$, the results obtained above are reduced to the ordinary case of foliations of codimension q. On the other hand, $B\Gamma$ can also be considered as the classifying space for foliated manifolds (M, F) with F of codimension q_1 and M of dimension q_2.

(3) The universal characteristic homomorphism

$$\widetilde{\Delta}_*: H(W_I) \cong H(W(\mathrm{gl}(q_1) \oplus \mathrm{gl}(d))_I) \to H(F\Gamma, R)$$

(resp. $\Delta'_*: H(W(\mathrm{gl}(q_1) \oplus \mathrm{gl}(d), \mathrm{SO}(q_1) \times \mathrm{SO}(d))_I) \to H(B\Gamma^+, R)$) for subfoliations of codimension (q_1, q_2) with trivialized normal bundle (resp. with oriented normal bundle) is constructed in an analogous way and results similar to those announced in Proposition 3.4 are obtained. Moreover, there is a commutative diagram

$$\begin{array}{ccccc}
H(WO_I) & \longrightarrow & H(W(\mathrm{gl}(q_1) \oplus \mathrm{gl}(d), \mathrm{SO}(q_1) \times \mathrm{SO}(d))_I) & \longrightarrow & H(W_I) \\
\downarrow{\scriptstyle \Delta_*} & & \downarrow{\scriptstyle \Delta'_*} & & \downarrow{\scriptstyle \widetilde{\Delta}_*} \\
H(B\Gamma, R) & \longrightarrow & H(B\Gamma^+, R) & \longrightarrow & H(F\Gamma, R)
\end{array}$$

with canonical horizontal maps.

4. The computation of the homomorphism p_*

In this section, using the notations of §3 in [6], we give the computation of the canonical homomorphism $p_*: H(WO_I) \to H(WO_{q_1}) \otimes H(WO_d)$ with $0 < q_1 < q_2$ and $d = q_2 - q_1 > 0$.

Proposition 4.1. *The cohomology classes in $H(WO_I)$ of the cocycles*

$$\begin{aligned} z_{(i,i',j,j')} &= y_{(i)} \wedge y'_{(i')} \otimes c_{(j)}c'_{(j')} \\ &= y_{i_1} \wedge \cdots \wedge y_{i_s} \wedge y'_{i'_1} \wedge \cdots \wedge y'_{i'_{s'}} \\ &\quad \otimes c_1^{j_1} \cdots c_{q_1}^{j_{q_1}} c_1'^{j'_1} \cdots c_d'^{j'_d} \in WO_I \end{aligned}$$

of the Vey basis (with $2p_1 = \deg c_{(j)}$, $2p_2 = \deg c'_{(j')}$, and $p = p_1 + p_2$) satisfying the conditions:

(1) $0 \le p_1 \le q_1$, $0 \le p_2 \le d$, $0 \le s \le [(q_1+1)/2]$, *and* $0 \le s' \le [(d+1)/2]$;
(2) $i_0 + p_1 \ge q_1 + 1$ *and* $i'_0 + p \ge q_2 + 1$;
(3) $i_0 \le j_0$ *and* $i'_0 \le j'_0$

are mapped under the homomorphism $p_*: H(WO_I) \to H(WO_{q_1}) \otimes H(WO_d)$ onto a basis of $\operatorname{Im} p_* \subset H(WO_{q_1}) \otimes H(WO_d)$.

Proof. Let C be the set of elements $z_{(i,i',j,j')}$ such that $0 \le p_1 \le q_1$, $0 \le p \le q_2$, $0 \le s \le [(q_1+1)/2]$, and $0 \le s' \le [(d+1)/2]$. Let C_a, C_b, and C_c be the sets of elements $z_{(i,i',j,j')} \in C$ satisfying the conditions:

(C_a) $i_0 > i'_0$, $i'_0 < j_0$, $i'_0 \le j'_0$, and $i'_0 + p \ge q_2 + 1$;
(C_b) $i_0 \le i'_0$, $i_0 \le j_0$, $i_0 \le j'_0$, $i_0 + p_1 < q_1 + 1$, and $i_0 + p \ge q_2 + 1$;
(C_c) $i_0 \le i'_0$, $i_0 \le j_0$, $i'_0 \le j'_0$, $i_0 + p_1 \ge q_1 + 1$, and $i'_0 + p \ge q_2 + 1$

respectively. In particular, we have $C_a \cap C_b = C_a \cap C_c = C_b \cap C_c = \varnothing$. It is clear that the elements $z_{(i,i',j,j')} \in C_a \cup C_b \cup C_c$ are cocycles and that the Vey basis of $H(WO_I)$ is given by the cohomology classes of these cocycles.

Clearly, we have $p_*[z_{(i,i',j,j')}] = 0$ for $z_{(i,i',j,j')} \in C_a \cup C_b \cup C_c$ such that $d < p_2 \le q_2$. It follows that $p_*[z_{(i,i',j,j')}] = 0$ for $z_{(i,i',j,j')} \in C_b$. On the other hand, consider the subspace $\widetilde{C}_a$ (resp. $\widetilde{C}_c$) of $H(WO_I)$ spanned by the classes of the cocycles $z_{(i,i',j,j')} \in C_a$ such that $0 \le p_2 \le d$ and $i_0 \le j_0$ (resp. of the cocycles $z_{(i,i',j,j')} \in C_c$ such that $0 \le p_2 \le d$). Then it is easy to see that the R-linear map

$$p_*|_{\widetilde{C}}: \widetilde{C} \to H(WO_{q_1}) \otimes H(WO_d)$$

is injective, where $\widetilde{C} = \widetilde{C}_a \oplus \widetilde{C}_c$.

Finally, for $z_{(i,i',j,j')} \in C_a$ with $0 \le p_2 \le d$ and $j_0 < i_0$, we obtain

$$\begin{aligned} p_*[z_{(i,i',j,j')}] &= [y_{(i)} \otimes c_{(j)}] \otimes [y'_{(i')} \otimes c'_{(j')}] \\ &= \sum_{t=1}^{s} (-1)^{t+1} p_*[y_{j_0} \wedge y_{i_1} \wedge \cdots \wedge \hat{y}_{i_t} \wedge \cdots \wedge y_{i_s} \wedge y'_{(i')} \otimes c_{i_t} \cdot \Phi \cdot c'_{(j')}], \end{aligned}$$

where $\Phi = c_1^{j_1} \cdots c_{j_0}^{j_{j_0}-1} \cdots c_{q_1}^{j_{q_1}}$ (with $p_*[z_{(i,i',j,j')}] = 0$ for $s = 0$); it follows that $p_*[z_{(i,i',j,j')}] \in p_*\widetilde{C}_a$. Whence $\operatorname{Im} p_* = p_*\widetilde{C}$. □

In a similar way, we obtain the following

Proposition 4.2. *An R-basis of $\operatorname{Ker} p_* \subset H(WO_I)$ is given by the union of the classes $[z_{(i,i',j,j')}]$ of the Vey basis of $H(WO_I)$ such that $d < p_2 \le q_2$ and the classes of the cocycles*

$$\begin{aligned} z'_{(i,i',j,j')} = z_{(i,i',j,j')} - \sum_{t=1}^{s} (-1)^{t+1} y_{j_0} \wedge y_{i_1} \wedge \cdots \wedge \hat{y}_{i_t} \\ \wedge \cdots \wedge y_{i_s} \wedge y'_{(i')} \otimes c_{i_t} \cdot \Phi \cdot c'_{(j')} \end{aligned}$$

such that $0 \le p_1 \le q_1$, $0 \le p_2 \le d$, $0 \le s \le [(q_1+1)/2]$, $0 \le s' \le [(d+1)/2]$, $i_0 > j_0 > i'_0$, $i'_0 \le j'_0$, *and* $i'_0 + p \ge q_2 + 1$, *where* $\Phi = c_1^{j_1} \cdots c_{j_0}^{j_{j_0}-1} \cdots c_{q_1}^{j_{q_1}}$ (*with* $z'_{(i,i',j,j')} = z_{(i,i',j,j')}$ *for* $s = 0$).

Let $\widetilde{C} \subset H(WO_I)$ be the subspace spanned by the cohomology classes of the cocycles $z_{(i,i',j,j')}$ considered in Proposition 4.1. Let $\widetilde{C}' \subset \widetilde{C}$ be a subspace with $\widetilde{C}' \neq 0$, such that $(p_*|_{\widetilde{C}})^{-1}(\operatorname{Ker}\widetilde{\Delta}'_*) \cap \widetilde{C}' = 0$ (evidently, the results of Kamber-Tondeur [17] imply that there is a subspace $\widetilde{C}' \subset \widetilde{C}$ of dimension $2^{[(q_1+1)/2]-1}(2^{[(d+1)/2]-1}+1)$ satisfying the property above), where $\widetilde{\Delta}'_* = \mu \circ (\Delta_{1*} \otimes \Delta_{0*})\colon H(WO_{q_1}) \otimes H(WO_d) \to H(B\Gamma_{q_1} \times B\Gamma_d, R)$ denotes the universal characteristic homomorphism for double foliations of codimension q_1 and d respectively, Δ_{1*} (resp. Δ_{0*}) being the universal characteristic homomorphism for foliations of codimension q_1 (resp. d), and μ the cohomology cross product (clearly, the homomorphism μ is injective). Then, from Propositions 3.4, 4.1, and 4.2 we obtain the following result.

Corollary 4.3. (i) *The R-linear maps*

$$\Delta_*|_{\widetilde{C}'}\colon \widetilde{C}' \to H(B\Gamma, R) \quad \text{and} \quad \phi^*|_{\Delta_*\widetilde{C}'}\colon \Delta_*\widetilde{C}' \to H(B\Gamma_{q_1} \times B\Gamma_d, R)$$

are injective.

(ii) *We have*

$$\Delta_* \operatorname{Ker} p_* \subset \operatorname{Ker}\phi^* \subset H(B\Gamma, R),$$
$$\operatorname{Im}\Delta_* = \Delta_*\widetilde{C} + \Delta_* \operatorname{Ker} p_* \subset \Delta_*\widetilde{C} + \operatorname{Ker}\phi^* \subset H(B\Gamma, R),$$
$$\Delta_*\widetilde{C} \cap \operatorname{Ker}\phi^* = \Delta_*((p_*|_{\widetilde{C}})^{-1}(\operatorname{Ker}\widetilde{\Delta}'_*)),$$
$$\Delta_*^{-1}(\operatorname{Ker}\phi^*) = (p_*|_{\widetilde{C}})^{-1}(\operatorname{Ker}\widetilde{\Delta}'_*) \oplus \operatorname{Ker} p_* \subset \widetilde{C} \oplus \operatorname{Ker} p_* = H(WO_I),$$
$$\operatorname{Im}\phi_1^*\Delta_{1*} \subset \Delta_*\widetilde{C}, \quad \text{and} \quad \operatorname{Im}\phi_1^*\Delta_{1*} \cap \operatorname{Ker}\phi^* = 0.$$

Corollary 4.4. *Let* u *be an element of* $H(WO_I)$. *Then* $u \in \Delta_*^{-1}(\operatorname{Ker}\phi^*)$ *if and only if* $\Delta_{*(F_1,F_2)}u = 0 \in H_{DR}(M)$ *for any manifold* M *and for any* (q_1, q_2)*-codimensional subfoliation* (F_1, F_2) *on* M *such that* $(F_1, F_2) = (F_1, F_1 \cap F_0)$ *with* F_0 *a d-codimensional foliation on* M.

Corollary 4.5. *For the universal Godbillon-Vey classes, we have*

(i) $\Delta_*[y'_1 \otimes c_1^j c_1'^{q_2-j}]$, $\Delta_*[y_1 \wedge y'_1 \otimes c_1^{j'} c_1'^{q_2-j'}] \in \Delta_* \operatorname{Ker} p_* \subset \operatorname{Ker}\phi^*$ *for* $0 \le j \le q_1$ *and* $0 \le j' \le q_1 - 1$. *In particular, for the universal Godbillon-Vey class* $\Delta_{2*}[y''_1 \otimes c_1''^{q_2}] \in H^{2q_2+1}(B\Gamma_{q_2}, R)$, *we have*

$$(\phi_2^* \circ \Delta_{2*})[y''_1 \otimes c_1''^{q_2}] = \sum_{j=0}^{q_1} \binom{q_2+1}{j} \Delta_*[y'_1 \otimes c_1^j c_1'^{q_2-j}] \in \Delta_* \operatorname{Ker} p_*.$$

(ii) $\Delta_*[y_1 \otimes c_1^{q_1}]$, $\Delta_*[y_1 \wedge y'_1 \otimes c_1^{q_1} c_1'^{d}] \in \Delta_*\widetilde{C} - \operatorname{Ker}\phi^*$.

Remark. In the same way, we can compute the canonical homomorphisms $\tilde{p}_*\colon H(W_I) \to H(W_{q_1}) \otimes H(W_d)$ and

$$p'_*\colon H(W(\mathrm{gl}(q_1) \oplus \mathrm{gl}(d), \mathrm{SO}(q_1) \times \mathrm{SO}(d))_I)$$
$$\to H(W(\mathrm{gl}(q_1), \mathrm{SO}(q_1))_{q_1}) \otimes H(W(\mathrm{gl}(d), \mathrm{SO}(d))_d).$$

Results similar to those announced above are obtained. It is clear that the homomorphism p'_* is given by $p'_*|_{H(WO_I)} = p_*$, $p'_* e_m = e_m$, and $p'_* e'_n = e'_n$, where $e_m \in I^{2m}(\mathrm{SO}(q_1))$ (resp. $e'_n \in I^{2n}(\mathrm{SO}(d))$) denotes the Pfaffian polynomial for $q_1 = 2m$ (resp. for $d = 2n$).

5. Examples and applications

In this section, using the examples of locally homogeneous subfoliations (with nontrivial characteristic classes) given in [7], we show that $\Delta_* \operatorname{Ker} p_* \subset H(B\Gamma, R)$ is nontrivial for $(q_1, q_2) = (d+1, 2d+1)$ with $d \geq 1$.

Let $H \subset G_2 \subset G_1 \subset \overline{G}$ be Lie groups, and $h \subset g_2 \subset g_1 \subset \overline{g}$ their Lie algebras. Assume that H is closed in $\overline{G}$. Let $\overline{\Gamma} \subset \overline{G}$ be a discrete subgroup acting properly discontinuously and without fixed points on $\overline{G}/H$, so that $M = \overline{\Gamma}\backslash\overline{G}/H$ is a manifold. A (q_1, q_2)-codimensional subfoliation (F_1, F_2) on M of the form $(F_1, F_2) = (F_{G_1}, F_{G_2})$ is called a locally homogeneous subfoliation, where $F_i = F_{G_i}$ is the locally homogeneous foliation of codimension $q_i = \dim \overline{g}/g_i$ on M, induced by the foliation on $\overline{G}$ defined by the right action of G_i, $i = 1, 2$. The computation of the characteristic homomorphism for locally homogeneous subfoliations has been described in [7].

Clearly, if G_1 and G_2 are connected or if H is connected, then $(F_1, F_2) = (F_{G_1}, F_{G_2})$ is a subfoliation with oriented normal bundle. Similarly, for $H = \{e\}$, $(F_1, F_2) = (F_{G_1}, F_{G_2})$ is a subfoliation with trivialized normal bundle.

Example 1. Let $\overline{G} = \mathrm{SL}(d+2)$, $G_1 = \mathrm{SL}(d+2, 1)_0$, $G_2 = \mathrm{SL}(d+2, 2)_0$, $H = \mathrm{SO}(d)$, and $\overline{T} \subset \mathrm{SL}(d+2)$ be a discrete uniform and torsion-free subgroup (with $d \geq 1$), where $\mathrm{SL}(d+2, 1)_0$ (resp. $\mathrm{SL}(d+2, 2)_0$) denotes the connected component of the group $\mathrm{SL}(d+2, 1)$ (resp. of the group $\mathrm{SL}(d+2, 2)$) of unimodular matrices of the form

$$\left(\begin{array}{c|c} \lambda & * \\ \hline 0 & A \end{array}\right)$$

with $A \in \mathrm{GL}(d+1)$ and $\lambda^{-1} = \det A$ (resp. of the form

$$\left(\begin{array}{c|cc} \lambda_1 & & * \\ \hline & \lambda_2 & * \\ \cline{2-3} 0 & 0 & B \end{array}\right)$$

with $B \in \mathrm{GL}(d)$, $\lambda_1, \lambda_2 \in \mathrm{GL}(1)$, and $\lambda_1^{-1} = \lambda_2 \cdot \det B$). Then, by virtue of Theorem 3.2 in [7], $M = \overline{\Gamma}\backslash\overline{G}/H$ is a connected compact orientable manifold and the canonical homomorphism $\gamma_*\colon H(\overline{g}, H) \to H_{\mathrm{DR}}(M)$ is injective. Furthermore, we have

$$H(\overline{g}, H) \cong \begin{cases} \bigwedge(\overline{y}_3, \overline{y}_5, \ldots, \overline{y}_{2n-1}, \overline{y}_{d+1}, \overline{y}_{d+2}) & \text{for } d = 2n-1, \\ \bigwedge(\overline{y}_3, \overline{y}_5, \ldots, \overline{y}_{2n-1}, \overline{y}_{d+1}, \overline{y}_{d+2}) \otimes R[e'_n]/(e'^2_n) & \text{for } d = 2n, \end{cases}$$

where the elements $\overline{y}_i$ are the relative suspensions of the Chern polynomials $\overline{c}_i \in I(\mathrm{SL}(d+2)) = R[\overline{c}_2, \overline{c}_3, \ldots, \overline{c}_{d+2}]$ and $e'_n \in I^{2n}(\mathrm{SO}(2n))$ is the Pfaffian polynomial.

On the other hand, consider the locally homogeneous subfoliation $(F_1, F_2) = (F_{G_1}, F_{G_2})$ of codimension $(q_1, q_2) = (d+1, 2d+1)$ with oriented normal bundle on M. Let

$$\Delta_{*(F_1, F_2)}\colon H(W(\mathrm{gl}(d+1) \oplus \mathrm{gl}(d), \mathrm{SO}(d+1) \times \mathrm{SO}(d))_I) \to H_{\mathrm{DR}}(M)$$

be the characteristic homomorphism of this subfoliation. Let $c_{(j)} = c_1^{j_1} \cdots c_{d+1}^{j_{d+1}}$ be a monomial of $\deg c_{(j)} = 2(d+1-k)$ and $c'_{(j')} = c_1'^{j'_1} \cdots c_d'^{j'_d}$ a monomial of $\deg c'_{(j')} = 2(d+k)$ with $0 \le k \le d+1$. Choose integers $t_0, t_1, \dots, t_{n-1}$ such that $0 \le t_s \le s$ for $s = 0, 1, \dots, n-1$, where $n = [(d+1)/2]$. Now, consider in $H(W(\mathrm{gl}(d+1) \oplus \mathrm{gl}(d), \mathrm{SO}(d+1) \times \mathrm{SO}(d))_I)$ the classes of the cocycles

$$z_{(i,j,j')} = y_1 \wedge y_{2i_1-1} \wedge \cdots \wedge y_{2i_{t_s}-1} \wedge y'_1 \wedge y'_{2i_{t_s+1}-1} \wedge \cdots \wedge y'_{2i_s-1} \otimes c_{(j)} c'_{(j')}$$

for $2 \le i_1 < \cdots < i_{t_s} < i_{t_s+1} < \cdots < i_s \le n$, $0 \le s \le n-1$, where $z_{(i,j,j')} = y_1 \wedge y'_1 \otimes c_{(j)} c'_{(j')}$ for $s = 0$, $z_{(i,j,j')} = y_1 \wedge y_{2i_1-1} \wedge \cdots \wedge y_{2i_s-1} \wedge y'_1 \otimes c_{(j)} c'_{(j')}$ for $s > 0$ and $t_s = s$, and $z_{(i,j,j')} = y_1 \wedge y'_1 \wedge y'_{2i_1-1} \wedge \cdots \wedge y'_{2i_s-1} \otimes c_{(j)} c'_{(j')}$ for $s > 0$ and $t_s = 0$. It is clear that the classes $[z_{(i,j,j')} \otimes \Phi]$ belong to the kernel of the canonical homomorphism

$$\begin{aligned} p'_*\colon & H(W(\mathrm{gl}(q_1) \oplus \mathrm{gl}(d), \mathrm{SO}(q_1) \times \mathrm{SO}(d))_I) \\ & \to H(W(\mathrm{gl}(q_1), \mathrm{SO}(q_1))_{q_1}) \otimes H(W(\mathrm{gl}(d), \mathrm{SO}(d))_d) \end{aligned}$$

for $1 \le k \le d+1$, where $\Phi = 1$ if $d = 2n-1$, and $\Phi = 1$ or e'_n if $d = 2n$. We have then the following result.

Theorem 5.1. *For $k = 0$ and $1 < k \le d+1$, we have the linearly independent secondary classes*

$$\Delta_{*(F_1, F_2)}[z_{(i,j,j')} \otimes \Phi] = \mu \cdot \gamma_*(\bar{y}_{2i_1-1} \wedge \cdots \wedge \bar{y}_{2i_s-1} \wedge \bar{y}_{d+1} \wedge \bar{y}_{d+2} \otimes \Phi)$$

with $2 \le i_1 < \cdots < i_{t_s} < i_{t_s+1} < \cdots < i_s \le n$, $0 \le s \le n-1$, $\Phi = 1$ if $d = 2n-1$, $\Phi = 1$ or e'_n if $d = 2n$, and

$$\mu = (-1)^{t_s}(d+2)(d+1)(a_{kk-1} - a_{kk}) \prod_{i=1}^{d+1} \binom{d+2}{i}^{j_i} \cdot \prod_{i=1}^{d} \binom{d+1}{i}^{j'_i} \ne 0,$$

where a_{kk-1}, $a_{kk} \in R$ (with $a_{kk-1} = 0$ for $k = 0$) are given by the polynomial

$$\begin{aligned} f_k(\lambda) &= \prod_{i=1}^{d} \left(\sum_{u=0}^{i} \left(\binom{i}{u} \Big/ \binom{d+1}{u} \right) \lambda^u \right)^{j'_i} \\ &= \sum_{v=0}^{d+k} a_{kv} \lambda^v \in R[\lambda], \qquad a_{kv} \in R. \end{aligned}$$

The corresponding classes then span the subspace

$$\begin{aligned} & \gamma_*(\mathrm{Ideal}(\bar{y}_{d+2} \wedge \bar{y}_{d+1})) \\ & = \begin{cases} \gamma_*((\bar{y}_{d+2} \wedge \bar{y}_{d+1}) \cdot \bigwedge(\bar{y}_3, \bar{y}_5, \dots, \bar{y}_{2n-1})) & \text{for } d = 2n-1, \\ \gamma_*((\bar{y}_{d+2} \wedge \bar{y}_{d+1}) \cdot \bigwedge(\bar{y}_3, \bar{y}_5, \dots, \bar{y}_{2n-1}) \otimes R[e'_n]/(e_n'^2)) & \text{for } d = 2n \end{cases} \end{aligned}$$

of $H_{\mathrm{DR}}(M)$ of dimension $2^{[d/2]}$. For $k = 1$, we have $\Delta_{*(F_1, F_2)}[z_{(i,j,j')} \otimes \Phi] = 0$.

Proof. It suffices to proceed as in the proof of Theorem 6.1 in [7]. It is easy to see that $a_{kk} = 1$ for $k = 0$. Thus we have only to show that $a_{kk-1} - a_{kk} \ne 0$ for $1 < k \le d+1$, and $a_{kk-1} - a_{kk} = 0$ for $k = 1$.

Now, using the vth derivative of $f_k(\lambda)$, $v = 0, 1, \dots, d+k$, by a direct computation of $a_{kv} > 0$ for $0 \le v \le d+k$, $1 \le k \le d+1$, we then obtain

$$a_{kv} < ((vd+k)/(vd+v)) a_{kv-1} \quad \text{for } 1 < v \le d+k, \ 1 \le k \le d+1.$$

It follows that $a_{kv} < a_{kv-1}$ for $\max(2, k) \le v \le d+k$, $1 \le k \le d+1$. Hence

$$a_{kk-1} - a_{kk} > 0 \quad \text{for } 1 < k \le d+1.$$

On the other hand, since $a_{k0} = 1$ and $a_{k1} = (d+k)/(d+1)$ for $1 \le k \le d+1$, we have $a_{k0} - a_{k1} = 0$ for $k = 1$. It follows that

$$\Delta_*[z_{(i,j,j')} \otimes \Phi] = 0 \quad \text{for } k = 1. \quad \square$$

Remark. It is clear that

$$a_{kk-1} - a_{kk} = (k-1)\binom{d+k}{d} \Big/ (d+1)^k \quad \text{for } c'_{(j')} = c'^{d+k}_1, \ 0 \le k \le d+1.$$

Theorem 4.6 in [6] and Theorem 5.1 imply the following

Corollary 5.2. *The subfoliation considered in Theorem 5.1 is not integrably homotopic to a subfoliation of codimension $(d+1, 2d+1)$ on M of the form $(F'_1, F'_1 \cap F'_0)$ with F'_0 a d-codimensional foliation on M.*

Let $(q_1, q_2) = (d+1, 2d+1)$ with $d \ge 1$. Consider the universal characteristic homomorphism

$$\Delta'_*: H(W(\mathrm{gl}(q_1) \oplus \mathrm{gl}(d), \mathrm{SO}(q_1) \times \mathrm{SO}(d))_I) \to H(B\Gamma^+, R)$$

(resp. $\Delta'_{i*}: H(W(\mathrm{gl}(q_i), \mathrm{SO}(q_i))_{q_i}) \to H(B\Gamma^+_{q_i}, R)$) for subfoliations of codimension (q_1, q_2) (resp. for foliations of codimension q_i, $i = 1, 2$) with oriented normal bundle, and the canonical homomorphisms $\phi'^*: H(B\Gamma^+, R) \to H(B\Gamma^+_{q_1} \times B\Gamma^+_d, R)$ and $\phi'^*_i: H(B\Gamma^+_{q_i}, R) \to H(B\Gamma^+, R)$, $i = 1, 2$. Then, by Theorem 6.1 in [7], Theorem 5.1, Propositions 3.4 and 4.2, and Corollary 4.3 (in the oriented case) we obtain the following result.

Theorem 5.3. *Let $z_{(i,j,j')}$ be cocycles as in Theorem 5.1 with $1 < k \le d+1$. Then the universal secondary characteristic classes*

$$\Delta'_*[z_{(i,j,j')} \otimes \Phi] \in \Delta'_* \operatorname{Ker} p'_* \subset \operatorname{Ker} \phi'^* \subset H(B\Gamma^+, R)$$

for all $2 \le i_1 < \cdots < i_{t_s} < i_{t_s+1} < \cdots < i_s \le n$, $0 \le s \le n-1$, $\Phi = 1$ if $d = 2n-1$ and $\Phi = 1$ or e'_n if $d = 2n$, are linearly independent. The corresponding classes then span a subspace $E \subset \Delta'_* \operatorname{Ker} p'_*$ of dimension $2^{[d/2]}$ satisfying $E \cap \operatorname{Im} \phi'^*_i \Delta'_{i*} = 0$, $i = 1, 2$.

Corollary 5.4. $\operatorname{Ker} \phi'^* \ne 0$. *Therefore, the canonical homomorphism ϕ'^*_1 is not surjective.*

Let $A \subset H(B\Gamma^+, R)$ be the subalgebra generated by all elements of $\operatorname{Im} \phi'^*_1 \Delta'_{1*} \cup \operatorname{Im} \phi'^*_2 \Delta'_{2*}$. Consider the subspace $E' \subset E$ of dimension 2^{n-1} spanned by the universal secondary characteristic classes $\Delta'_*[z_{(i,j,j')} \otimes \Phi]$ given in Theorem 5.3 with $\Phi = 1$ for $d = 2n-1$ and $\Phi = e'_n$ for $d = 2n$. Then we have the following corollary.

Corollary 5.5. $E' \cap A = 0$.

Similarly, applying Theorem 6.1 in [7], Theorem 5.1, Propositions 3.4 and 4.2, and Corollary 4.3, we obtain the following

Corollary 5.6. *There is a subspace* $\widetilde{N} \subset \operatorname{Im} \Delta'_*$ *of dimension* $2^{[d/2]}$, *spanned by universal secondary characteristic classes, such that* $\widetilde{N} \cap A = 0$ *and* $\widetilde{N} \cap \operatorname{Ker} \phi'^* = 0$.

Example 2. Let $\overline{G} = \mathrm{SL}(d+2)$, $G_1 = \mathrm{SL}(d+2, 1)$, $G_2 = \mathrm{SL}(d+2, 2)$, $H = \mathrm{O}(d)$, and $\overline{\Gamma} \subset \mathrm{SL}(d+2)$ be as in Example 1 (with $d \geq 1$). Consider the locally homogeneous subfoliation $(F_1, F_2) = (F_{G_1}, F_{G_2})$ of codimension $(q_1, q_2) = (d+1, 2d+1)$ on $M = \overline{\Gamma} \backslash \overline{G}/H$ (whose normal bundle is not necessarily orientable). Then, by Theorem 6.2 in [7], Theorems 5.1 and 5.3, Propositions 3.4 and 4.2, and Corollary 4.3 we obtain the following result.

Theorem 5.7. *Let* $(q_1, q_2) = (d+1, 2d+1)$ *with* $d \geq 1$. *Let* $z_{(i,j,j')}$ *be cocycles as in Theorem* 5.3. *Then the universal secondary characteristic classes*

$$\Delta_*[z_{(i,j,j')}] \in \Delta_* \operatorname{Ker} p_* \subset \operatorname{Ker} \phi^* \subset H(B\Gamma, R)$$

for all $2 \leq i_1 < \cdots < i_{t_s} < i_{t_s+1} < \cdots < i_s \leq n = [(d+1)/2]$, $0 \leq s \leq n-1$, *are linearly independent. The corresponding classes then span a subspace* $E \subset \Delta_* \operatorname{Ker} p_*$ *of dimension* 2^{n-1} *satisfying* $E \cap \operatorname{Im} \phi_i^* \Delta_{i*} = 0$, $i = 1, 2$. *For* $d = 2n - 1$, *we have* $E \cap A = 0$, *where* $A \subset H(B\Gamma, R)$ *denotes the subalgebra generated by all elements of* $\operatorname{Im} \phi_1^* \Delta_{1*} \cup \operatorname{Im} \phi_2^* \Delta_{2*}$. *Furthermore, for* $d = 2n-1$, *there is a subspace* $\widetilde{N} \subset \operatorname{Im} \Delta_*$ *of dimension* 2^{n-1}, *spanned by universal secondary characteristic classes, such that* $\widetilde{N} \cap A = 0$ *and* $\widetilde{N} \cap \operatorname{Ker} \phi^* = 0$.

Corollary 5.8. $\operatorname{Ker} \phi^* \neq 0$. *It follows that the canonical homomorphism*

$$\phi_1^* \colon H(B\Gamma_{q_1}, R) \to H(B\Gamma, R)$$

is not surjective.

Example 3. Let $\overline{G}$, G_1, and G_2 be as in Example 2, $H = \{e\}$, and $\overline{\Gamma} \subset \mathrm{SL}(d+2)$ a discrete uniform subgroup (with $d \geq 1$). Consider the locally homogeneous subfoliation $(F_1, F_2) = (F_{G_1}, F_{G_2})$ of codimension $(q_1, q_2) = (d+1, 2d+1)$ with trivialized normal bundle on $M = \overline{\Gamma} \backslash \overline{G}$. Now, let $c_{(j)}$ and $c'_{(j')}$ be as in Theorem 5.1 with $1 < k \leq d+1$. Let $t_0, t_1, \ldots, t_{d-1}$ be integers such that $0 \leq t_s \leq s$ for $s = 0, 1, \ldots, d-1$. Consider in $H(W_I)$ the cohomology classes of the cocycles

$$z_{(i,j,j')} = y_1 \wedge y_{i_1} \wedge \cdots \wedge y_{i_{t_s}} \wedge y'_1 \wedge y'_{i_{t_s+1}} \wedge \cdots \wedge y'_{i_s} \otimes c_{(j)} c'_{(j')}$$

with $2 \leq i_1 < \cdots < i_{t_s} < i_{t_s+1} < \cdots < i_s \leq d$, $0 \leq s \leq d-1$, where $z_{(i,j,j')} = y_1 \wedge y'_1 \otimes c_{(j)} c'_{(j')}$ for $s = 0$, $z_{(i,j,j')} = y_1 \wedge y_{i_1} \wedge \cdots \wedge y_{i_s} \wedge y'_1 \otimes c_{(j)} c'_{(j')}$ for $s > 0$ and $t_s = s$, and $z_{(i,j,j')} = y_1 \wedge y'_1 \wedge y'_{i_1} \wedge \cdots \wedge y'_{i_s} \otimes c_{(j)} c'_{(j')}$ for $s > 0$ and $t_s = 0$. Then, by a technique analogous to that used in the proof of Theorems 5.1 and 5.3 but from more elementary computations we obtain

Theorem 5.9. *Let* $(q_1, q_2) = (d+1, 2d+1)$ *with* $d \geq 1$. *Then the universal secondary characteristic classes*

$$\widetilde{\Delta}_*[z_{(i,j,j')}] \in \widetilde{\Delta}_* \operatorname{Ker} \tilde{p}_* \subset \operatorname{Ker} \tilde{\phi}^* \subset H(F\Gamma, R)$$

for all $2 \leq i_1 < \cdots < i_{t_s} < i_{t_s+1} < \cdots < i_s \leq d$, $0 \leq s \leq d-1$, *are linearly independent. The corresponding classes then span a subspace* $E \subset \widetilde{\Delta}_* \operatorname{Ker} \tilde{p}_*$ *of dimension* 2^{d-1} *satisfying* $E \cap \operatorname{Im} \tilde{\phi}_i^* \widetilde{\Delta}_{i*} = 0$, $i = 1, 2$.

Corollary 5.10. *For* $(q_1, q_2) = (d+1, 2d+1)$ *with* $d \geq 1$, *the canonical homomorphism* $\tilde{\phi}^*: H(F\Gamma, R) \to H(F\Gamma_{q_1} \times F\Gamma_d, R)$ *is not injective. Hence, the canonical homomorphism* $\tilde{\phi}_1^*: H(F\Gamma_{q_1}, R) \to H(F\Gamma, R)$ *is not surjective.*

Remark. Let (q_1, q_2) be a couple of integers with $0 < q_1 < q_2$. It follows from [11, 14, 15] that the canonical homomorphisms ϕ^*, ϕ'^*, $\tilde{\phi}^*$, ϕ_2^*, $\phi_2'^*$ and $\tilde{\phi}_2^*$ are not surjective.

Proposition 5.11. *Let* (q_1, q_2) *be a couple of integers with* $0 < q_1 < q_2$, $(q_1, q_2) \neq (2m-1, 2m)$, *and* $(q_1, q_2) \neq (1, 2m)$. *Then the canonical homomorphism* $\phi_2^*: H(B\Gamma_{q_2}, R) \to H(B\Gamma, R)$ *is not injective.*

Proof. Consider in $H(WO_{q_2})$ the cohomology class of a monomial cocycle of the form

$$z = y_1'' \wedge y_{2q_2'-1}'' \otimes c_{(j)}'' = y_1'' \wedge y_{2q_2'-1}'' \otimes c_1''^{j_1} \cdots c_{q_2}''^{j_{q_2}}$$

with $\deg c_{(j)}'' = 2q_2$, where $q_2' = [(q_2+1)/2] \geq 2$. Then, from [7] it follows that $\Delta_{2*}[z] \neq 0 \in H(B\Gamma_{q_2}, R)$. On the other hand, by virtue of Corollary 5.2 in [7], the cohomology class $[z]$ is in the kernel of the canonical homomorphism $W(d\rho_2)^*: H(WO_{q_2}) \to H(WO_I)$. Whence, in view of Proposition 3.4, we have $\Delta_{2*}[z] \in \operatorname{Ker} \phi_2^*$. □

Remarks. (1) It is clear that $\Delta_{2*}[1 \otimes c_{q_2}''] \in \operatorname{Ker} \phi_2^*$ for $(q_1, q_2) = (2n-1, 2m)$ with $0 < q_1 < q_2$. Unfortunately, we have been unable to prove that $\Delta_{2*}[1 \otimes c_{q_2}''] \neq 0$.

(2) A geometric interpretation for nontrivial elements of the kernel of the canonical homomorphism $W(d\rho_2)^*$ has been given in [7] (see also [5]).

(3) In a similar way, we show that the canonical homomorphisms

$$\phi_2'^*: H(B\Gamma_{q_2}^+, R) \to H(B\Gamma^+, R)$$

and

$$\tilde{\phi}_2^*: H(F\Gamma_{q_2}, R) \to H(F\Gamma, R)$$

are not injective for $0 < q_1 < q_2$ Evidently, $\Delta_*'[1 \otimes e_m''] \in \operatorname{Ker} \phi_2'^*$ is not zero for $(q_1, q_2) = (2n-1, 2m)$ with $0 < q_1 < q_2$, where $e_m'' \in I^{2m}(\mathrm{SO}(q_2))$ is the Pfaffian polynomial for $q_2 = 2m$. Analogously, it is easily shown that the element $\tilde{\Delta}_*[y_1'' \wedge y_{q_2}'' \otimes c_{(j)}''] \in \operatorname{Ker} \tilde{\phi}_2^*$ is not zero for $\deg c_{(j)}'' = 2q_2$ with $0 < q_1 < q_2$.

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